Hardy spaces of the real Beltrami equation

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We introduce Hardy spaces of solutions to the so-called real Beltrami equation in the disk: $\overline{\partial} f = \nu(z)\overline{\partial f}$, where $-1+\varepsilon < \nu < 1-\varepsilon$ and ν is Lipschitz continuous. Dwelling on some work by Bers and Nirenberg on pseudo-analytic functions, we prove the L^p boundedness of the conjugation operator mapping u to v if $f = u + iv$ on the circle, and f has real mean. We also show the density of such functions on strict subarcs of the circle. This allows us to consider bounded extremal problems in such classes of functions. A motivation for such a study comes from the fact that the compatibility condition for $f = u + iv$ to solve the Beltrami equation is that $div(\sigma\nabla u) = 0$ where $\sigma = (1-\nu)/(1+\nu)$. This way, extremal problems arising for solutions to diffusion equations cans be recast in terms of pseudo-analytic functions. We exemplify this in the case of an inverse boundary problem arising in plasma control.

Orthogonal polynomials and the interlacing of zeros

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It is a well-known classical result that the zeros of orthogonal polynomials of adjacent degree are interlacing. We discuss the extent to which the interlacing of zeros can be proved in many different situations where orthogonal polynomials, particularly classical orthogonal polynomials, are involved.

The zeros of polynomials of the same or adjacent degree from different orthogonal sequences may or may not interlace and we give proofs (or counter-examples where appropriate) for the one-parameter families of Laguerre and Gegenbauer polynomials, as well as the two-parameter family of Jacobi polynomials. In these cases, the different sequences are generated by allowing the parameter(s) to vary continuously and/or in integer steps.

We review related results for the interlacing of zeros of linear combinations of classical orthogonal polynomials, including those that arise as a result of quasi-orthogonality.

We conclude with a discussion of an open question raised by F Marcellan in 2007 concerning the conditions under which linear combinations of orthogonal polynomials from distinct orthogonal sequences are themselves orthogonal.

Biorthogonal polynomials and the coupled random matrix model

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Statistical properties of eigenvalues of random matrices taken from a probability measure

$$\frac{1}{Z_n} \exp\left(-n \operatorname{Tr} V(M)\right) dM, \qquad V \text{ is a polynomial,}$$

defined on the space of $n \times n$ Hermitian matrices M can be fully analyzed using orthogonal polynomials. In this way an almost complete picture has arisen about the possible limiting eigenvalue behaviors as $n \to \infty$, both in the macroscopic and microscopic regimes.

The coupled random matrix model is a probability measure

$$\frac{1}{Z_n} \exp\left(-n \operatorname{Tr}(V(M_1) + W(M_2) - \tau M_1 M_2)\right) dM_1 dM_2$$

defined on pairs (M_1, M_2) of $n \times n$ Hermitian matrices. Here V and W are two polynomial potentials and $\tau > 0$ is a coupling constant. The model is of interest in $2D$ quantum gravity where it is used to construct generating functions for the number of bicolored graphs on surfaces.

The role of orthogonal polynomials is now taken over by two sequences of polynomials $\left(p_j^{(n)}\right)_j$ and $\left(q_k^{(n)}\right)_k$ that satisfy the biorthogonality condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_j^{(n)}(x) q_k^{(n)}(y) e^{-n(V(x)+W(y)-\tau xy)} dx dy = \delta_{j,k}.$$

Statistical properties of the eigenvalues of M_1 are described by the polynomials $p_j^{(n)}$. Despite many contributions in the physics literature, the limiting behavior is not fully understood in the mathematical sense.

In the talk I will discuss an approach to the simplest non-trivial case $W(y) = \frac{1}{4}y^4$, which involves the following steps:

- A characterization of the polynomials $p_j^{(n)}$ as multiple orthogonal polynomials, which leads to the formulation of a 4×4 matrix valued Riemann-Hilbert problem.
- An energy minimization problem for a triple of measures (μ_1, μ_2, μ_3) where μ_1 is the asymptotic zero distribution of the polynomials $p_n^{(n)}$ as well as the limiting eigenvalue distribution of M_1.
- The steepest descent analysis of the Riemann-Hilbert problem.

This is joint work with Maurice Duits.

Rational interpolation to exponential-like functions on elliptic lattices

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A function is called exponential-like with respect to a difference operator if it satisfies

$$Df(x) = a[f(\psi(x)) + f(\phi(x))],$$

where the (divided) difference operator is

$$Df(x) = [f(\psi(x)) - f(\phi(x))]/[\psi(x) - \phi(x)].$$

The functions ϕ and ψ define the setting of the theory, from the most elementary choice $(x, x+h)$ to forms $R(x)$ plus or minus square root of $S(x)$, where R and S are rational functions of degrees up to 2 and 4. Remark that the difference equation is a symmetric combination of the two conjugate algebraic functions ϕ and ψ. The difference equation is also a recurrence relation on a lattice built from $y(n) = \phi(x(n))$, $y(n+1) = \psi(x(n))$, from which $x(n+1)$ is found through $y(n+1) = \phi(x(n+1))$. When the degrees of$R$ and S are 2 and 4, we get a so-called elliptic lattice, or grid, as $x(n)$ and $y(n)$ appear to be elliptic functions of n (Baxter, Spiridonov, Zhedanov). The exponential-like function of above is interpolated on such a lattice $y(0), y(1), ...$ by rational functions with poles on a well-chosen sequence $y'(0), y'(1), ...$

Non-intersecting squared Bessel paths and multiple orthogonal polynomials for modified Bessel weights

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We study a model of n non-intersecting squared Bessel processes in the confluent case: all paths start at time $t = 0$ at the same positive value $x = a$, remain positive, and are conditioned to end at time $t = T$ at $x = 0$. In the limit $n \to \infty$, after appropriate rescaling, the paths fill out a region in the tx-plane that we describe explicitly. In particular, the paths initially stay away from the hard edge at $x = 0$, but at a certain critical time t^* the smallest paths hit the hard edge and from then on are stuck to it. For $t \neq t^*$ we obtain the usual scaling limits from random matrix theory, namely the sine, Airy, and Bessel kernels. A key fact is that the positions of the paths at any time t constitute a multiple orthogonal polynomial ensemble, corresponding to a system of two modified Bessel-type weights. As a consequence, there is a 3×3 matrix valued Riemann-Hilbert problem characterizing this model, that we analyze in the large n limit using the Deift-Zhou steepest descent method. There are some novel ingredients in the Riemann-Hilbert analysis that are of independent interest.

This is a joint work with A.B.J. Kuijlaars and F. Wielonsky

Interpolation and design: from polynomials to Chebyshevian splines

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Let us consider an $(N+1)$-dimensional space $\mathcal{E}$ of sufficiently differentiable functions. Roughly speaking, **design** in the space $\mathcal{E}$ refers to the possibility of drawing a curve with a prescribed approximate shape, fixed by a polygonal line with $(N+1)$ vertices, called its *control polygon*. The curve is expected to mimic its control polygon of which it should be a smooth version. Here,

interpolation in the space $\mathcal{E}$ refers to *Hermite interpolation*, *i.e.*, the possibility of determining a unique element in $\mathcal{E}$ with $(N+1)$ prescribed values for this element itself and its successive derivatives at given points, called *interpolation abscissæ*. The present talk surveys the strong links existing between the latter two topics which may a priori seem completely disconnected.

As is well-known, both interpolation and design are possible when $\mathcal{E}$ is the degree N polynomial space $\mathcal{P}_N$. It is the fact that any non-zero polynomial of degree at most N vanishes at most N times, counting multiplicities, which permits interpolation in $\mathcal{P}_N$. It is the presence of the Bernstein basis

$$B_i^N(x) := \left(\frac{x-a}{b-a}\right)^i \left(\frac{b-x}{b-a}\right)^{N-i}, \quad 0 \le i \le N,$$

with all its interesting properties, which makes design possible in $\mathcal{P}_N$. Now, the actual underlying reason explaining both existence and properties of this special basis is the presence of *blossoms* in the space $\mathcal{P}_N$. Indeed, any polynomial $F \in \mathcal{P}_N$ uniquely blossoms into a function f of N variables meeting the following three requirements:

(i) f is symmetric,

(ii) f is affine in each variable,

(iii) f gives F by restriction to the diagonal.

The function f is called the blossom of F. Blossoms are wonderful tools for design, for they make the description of all design algorithms extremely simple and elegant.

Nevertheless, as degree grows, both polynomial interpolation and polynomial design quickly turn to be only theoretical possibilities. Indeed, the mimicking of the control polygon becomes not good enough, while interpolating polynomials may have nonsensical behaviour. For this reason, when N is not small, it is way more reasonable to replace the polynomial space $\mathcal{P}_N$ by an $(N+1)$-dimensional space of *polynomial splines*, *i.e.*, functions which are piecewise polynomials, two consecutive pieces joining with precribed smoothness at the corresponding *knot*. For instance, the most commonly used ones, *cubic splines*, are C^2 and have pieces of degree 3.

Polynomial spline spaces are excellent for design due to the presence of *B-spline bases* and to their properties. Among them, let us mention the fact they have small supports, which permits a local control of the spline curves. Now, again their existence as well as their interesting properties are actually due to the underlying presence of blossoms: each spline S with degree n pieces uniquely blossoms into a function s of n variables, called its blossom, meeting the same requirements as previously, except that it is defined only on a restricted set of n-tuples.

In a polynomial spline space, interpolation is possible only provided that the interpolating abscissæ and the knots of the spline space interlace according to the so-called *Schoenberg-Whitney conditions*. Again, the fact that interpolation under Schoenberg-Whitney conditions is possible is related to the existence of B-spline bases. Therefore, one can say that it is implicitly related to the existence of blossoms.

Although way much better than the polynomial case, interpolation by polynomial splines still presents some flaws: unfortunately, we may still have undesired oscillations, in particular in case there is a jump in the data. This is often referred to as *Gibbs phenomenon*. In order to make up for this inconvenience, a classical idea consists in introducing *shape parameters*, *i.e.*, some parameters on which we can play to improve the interpolating curve (or function) where necessary while keeping its general shape. In spline spaces, there are two main ways to generate such parameters. One can

1- either insert *connection matrices* at the knots, the usual smoothness being replaced by a geometrical one – this generates *geometrically continuous polynomial splines*;

2- or replace the polynomial space in which splines have their sections by a *Chebyshev space* of the same dimension. For instance one can replace the polynomial space $\mathcal{P}_3$ by the space spanned by the four functions $1, x, \cosh x, \sinh x$.

Our $(N+1)$-dimensional initial space $\mathcal{E}$ is a Chebyshev space if any non-zero element vanishes at most N times, counting multiplicities. Such spaces are thus exactly the spaces in which interpolation is possible. What about design? It is possible in an $(N+1)$-dimensional space $\mathcal{E}$ which contains constants if and only if it possesses Bernstein type bases, or if and only if it possesses blossoms, now defined in a geometrical way by means of intersections of osculating flats : any $F \in \mathcal{E}$ then blossoms into a function f of n variables satisfying the two properties (i) and (iii) above, and a modified poperty (ii). Interpolation and design are strongly linked: design is possible in $\mathcal{E}$ iff interpolation is possible in the space $D\mathcal{E}$, *i.e.*, iff the space $D\mathcal{E}$ is a Chebyshev space. In any associate spline space $\mathcal{S}$ with ordinary smoothness at the knots, blossoms automatically exist. This permits both design in $\mathcal{S}$ (existence of B-spline type bases) and interpolation under Schoenberg-Whitney conditions.

Mixing the two ideas above, one can eventually consider the general framework of splines with sections in different Chebyshev spaces, and with connection matrices at the knots. Unfortunately, blossoms do not always exist in a space $\mathcal{S}$ of such splines supposed to contain constants. Nevertheless, their existence is the necessary and sufficient condition which makes possible either design in $\mathcal{S}$ or interpolation under Schoenberg-Whitney conditions in $D\mathcal{S}$.

Unfortunately, it is quite difficult to find whether or not blossoms exist in such a general context. Sufficient conditions do exist, but they are sometimes far too restrictive. In the interesting case of splines with simple knots and sections in four-dimensional spaces, we managed to find explicit necessary and sufficient conditions for existence of blossoms. We use them to illustrate how to use the many shape parameters we have at our disposal either for design, or for interpolation, or - why not - for *interpolating design*.

Well and Ill Posed Inversion Problems for Matrix Algebras

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Given a matrix or an operator T with eigenvalues λ_j, $j \geq 1$, we say that the inversion problem for functions (polynomials) in T is well posed if the norm of $f(T)^{-1}$ can be bounded in terms of $min_j|f(\lambda_j)| = \delta > 0$ for every f such that $\|f(T)\| \leq 1$.

(1) We give a criterion for such a well posedness in terms of the so-called (Carleson-like) Weak Embedding Property for $\sigma = (\lambda_j)$ and give many examples of the spectra satisfying (or not) this condition (joint result with P.Gorkin and R.Mortini).

(2) Moreover, given a constant $0 \leq \delta_1 \leq 1$ we show that there exist (infinite) spectra σ such that the above inversion problem is well posed for all δ with $\delta_1 < \delta \leq 1$ and ill posed for all δ with $0 < \delta < \delta_1$ (joint result with V.Vasyunin).

(3) Finally, we use these results for disproving an analog of the paving conjecture for Hilbert space unconditional block-bases (for Hilbert space unconditional bases, this conjecture is equivalent to the famous Kadison-Singer problem).

Expansion of Euler constant in terms of odd Zeta values

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An integral representation for Euler's constant is

$$\gamma = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) e^{-t}\, dt. \tag{1}$$

If we substitute the integrand $\left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right)$ by some rational fraction of degre $(n, m), (n \geq m-1)$ involving Padé approximants, then γ is approached by a linear combination of logarithm function $\log(k+1)$ and Zeta values, $\zeta(i), i = 1, n$, where k is some parameter.

If $k = 0$ and $n = m-1$, this is reduced to a linear combination of odd zeta values with coefficients in $\mathbf{Q}$. Arithmetical properties of this expansion are proved.

Moreover, from the properties of the remainder, an integral representation of the Euler constant is given.

The same method applied to another expression of γ permit to recover Sondow or Pilehrood formulas.

Interpolation fonctionnelle et approximation rationnelle de constantes classiques

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Interpolation series theory (i.e., expansion of entire functions in series of polynomials where the roots of the polynomials belong to a fixed set of $\mathbb{C}$) played an important role in diophantine approximation at the beginning of the 20th century. In particular, it was used by Pólya [6] when he proved that the function 2^z is the (non polynomial) entire function of smallest growth which sends $\mathbb{N}$ in $\mathbb{Z}$. The transcendence of e^α for any algebraic number $\alpha \neq 0$ (Hermite-Lindemann theorem) was also obtained by Siegel [8] by expanding $\exp(z)$ at suitable interpolation points.

Interpolation methods were crucial in Gel'fond's proof the transcendence of e^π (see [3]): this was a first step towards the proof of Hilbert's 7th problem that α^β is transcendental when

α, β are algebraic numbers, with $\alpha \neq 0, 1$ and β irrational. Despite some works by Boehle [2], Kuzmin [4] and Siegel [8] for example, interpolation methods were replaced by more powerful (but less explicit) methods based on auxiliary functions contructed using Siegel's lemma.

The aim of my talk is to report on my recent work [7], in which I show how another kind of interpolation process can be used in irrationality theory. More precisely, I show that the irrationality of $\log(2), \zeta(2)$ and $\zeta(3)$ (Apéry's theorem [1]) can be obtained by expanding the Hurwitz zeta function $\zeta(s, z) = \sum_{k=1}^{\infty} 1/(k+z)^s$ or related functions in interpolation series of rational functions (not only polynomials). Such an interpolation process was first studied by René Lagrange [5] in 1935 when the degree of the numerators and denominators of the rational summands are essentially equal. For example, using certain of his formulae, I proved the following result.

Theorem 1 (Rivoal, 2006)**.** *For all $z \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, we have that*

$$\zeta(2, z) = \sum_{n=0}^{\infty} A_{2n} \frac{(z-n+1)_n^2}{(z+1)_n^2} + \sum_{n=0}^{\infty} A_{2n+1} \frac{(z-n+1)_n^2}{(z+1)_n^2} \frac{z-n}{z+n+1},$$

where $A_0 = \zeta(2)$ and, for all $n \geq 0$,

$$A_{2n+1} = \frac{2n+1}{2\pi i} \int_{\mathscr{C}_n} \frac{(x+1)_n^2}{(x-n)_{n+1}^2} \zeta(2, x) \, \mathrm{d}x \in \mathbb{Q}\zeta(3) + \mathbb{Q}$$

and

$$A_{2n+2} = \frac{2n+2}{2\pi i} \int_{\mathscr{C}_n} \frac{(x+1)_n^2}{(x-n)_{n+1}^2} \frac{x+n+1}{x-n-1} \zeta(2, x) \, \mathrm{d}x \in \mathbb{Q}\zeta(3) + \mathbb{Q}.$$

The curve $\mathscr{C}_n$ encloses the points $0, 1, \ldots, n$ but none of the poles of $\zeta(2, z)$.

(By definition, $(u)_0 = 1$ and $(u)_m = u(u+1)\cdots(u+m-1)$ for $m \geq 1$.) The irrationality of $\zeta(3)$ is a corollary of this theorem. Indeed, by the residue theorem, it is easy to compute explicitely the coefficient A_n and to deduce that

$$d_n^3 A_n = u_n \zeta(3) - v_n \in \mathbb{Z}\zeta(3) + \mathbb{Z}$$

where $d_n = \operatorname{lcm}(1, 2, \ldots, n)$. Furthermore, from the integral representation of A_n, we obtain that

$$\limsup_{n \to +\infty} (d_n^3 A_n)^{1/n} \leq e^3 (\sqrt{2} - 1)^4 < 1.$$

Since $\zeta(2, z)$ is not a rational function of z, we necessarily have $A_n \neq 0$ for infinitely many n and the irrationality of $\zeta(3)$ is proved.

One can also obtain the irrationality of $\log(2)$ by René Lagrange's interpolation but I don't know if it is possible to obtain that of $\zeta(2)$ by these means. Instead, I found new interpolation formulae which enabled me to use rational functions with unequal degrees for the numerators and denominators. The irrationality of $\zeta(2)$ is then a consequence of the following theorem. By a slight abuse of notations, let

$$\zeta(1, z) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right).$$

Theorem 2 (Rivoal, 2006). *For all $z \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, we have*

$$\zeta(1, z) = \sum_{n=0}^{\infty} A_n \frac{(z-n+1)_n^2}{(z+1)_n} + \sum_{n=0}^{\infty} B_n \frac{(z-n+1)_n^2}{(z+1)_n} \frac{z-n}{z+n+1}$$

where $A_0 = B_0 = 0$ and, for all $n \geq 1$,

$$A_n = \frac{1}{2\pi i} \int_{\mathscr{C}_n} \frac{(x+1)_n (x-n)}{(x-n)_{n+1}^2} \zeta(1, x) \, \mathrm{d}x \in \mathbb{Q}\zeta(2) + \mathbb{Q}$$

and

$$B_n = \frac{2n}{2\pi i} \int_{\mathscr{C}_n} \frac{(x+1)_n}{(x-n)_{n+1}^2} \zeta(1, x) \, \mathrm{d}x \in \mathbb{Q}\zeta(2) + \mathbb{Q}.$$

The curve $\mathscr{C}_n$ encloses the points $0, 1, \ldots, n$ but none of the poles of $\zeta(1, z)$.

Bibliographie

[1] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque **61** (1979), 11–13.

[2] K. Boehle, *Über die Transzendenz von Potenzen mit algebraischen Exponenten (Verallgemeinerung eines Satzes von A. Gelfond)*, Math. Ann. **108** (1933), 56–74.

[3] A. O. Gel'fond, *Sur les nombres transcendants*, C. R. Acad. Sci. de Paris **189** (1929), 1224–1226.

[4] R. O. Kuzmin, *On a new class of transcendental numbers*, en russe, Izv. Akad. Nauk SSSR 3 (1930), 583–597.

[5] R. Lagrange, *Mémoire sur les séries d'interpolation*, Acta Math. **64** (1935), 1–80.

[6] G. Pólya, *Über ganzwertige ganze Funktionen*, Palermo Rend. **40** (1916), 1–16 (1916).

[7] T. Rivoal, Applications arithmétiques de l'interpolation lagrangienne, 24 pages, à paraître dans International Journal of Number Theory.

[8] C. Siegel, *Über die Perioden elliptischer Funktionen*, J. reine angew. Math. **167** (1932), 62–69.

Bergman Polynomials on Archipelagos

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We investigate the asymptotic behavior of polynomials orthonormal over regions G in the complex plane with respect to area measure (Bergman poly- nomials) in the case when $G = \cup_{j=1}^{N} G_j$ consists of the finite union of $N \geq 2$ mutually exterior Jordan domains. Our results concern the limiting behavior of the zeros of these polynomials as well as fine estimates for their leading coefficients. We also discuss a technique for the reconstruction of the component domains G_j from the area moments over G.

(Joint work with B. Gustafsson, M. Putinar and N. Stylianopoulos.)

Calculating Rational Best Approximants on $(-\infty, 0]$

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In many applications one needs rational approximations on the negative axis $\mathbb{R}_-$ of the exponential function or a function of similar type. In our talk we consider rational best approximants $r^*_{n,n+k} = r^*_{n,n+k}(f, \mathbb{R}_-; \cdot) \in R_{n,n+k}$ of a given function f on $\mathbb{R}_-$ in the uniform norm.

After a short review of characteristic properties of such approximants (the $'1/9'-$problem and related asymptotic considerations), we concentrate on numerical methods for their calculation. In the literature one finds two approaches for practical use: One is based on AAK approximation after the problem has been transformed from $\mathbb{R}_-$ onto the unit circle, and the other one has the Remez algorithm as its core piece.

We will describe a new variant of the algorithm. One of its main features is the exploitation of structural properties of the rational best approximants $r^*_{n,n+k}$, another one is the use of specific knowledge of the asymptotic behaviour of the error function.